

Fluid Mechanics Lecture Note – CVE 311

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LESSON ONE

Derivation of Euler and Bernoulli Equation of Motion

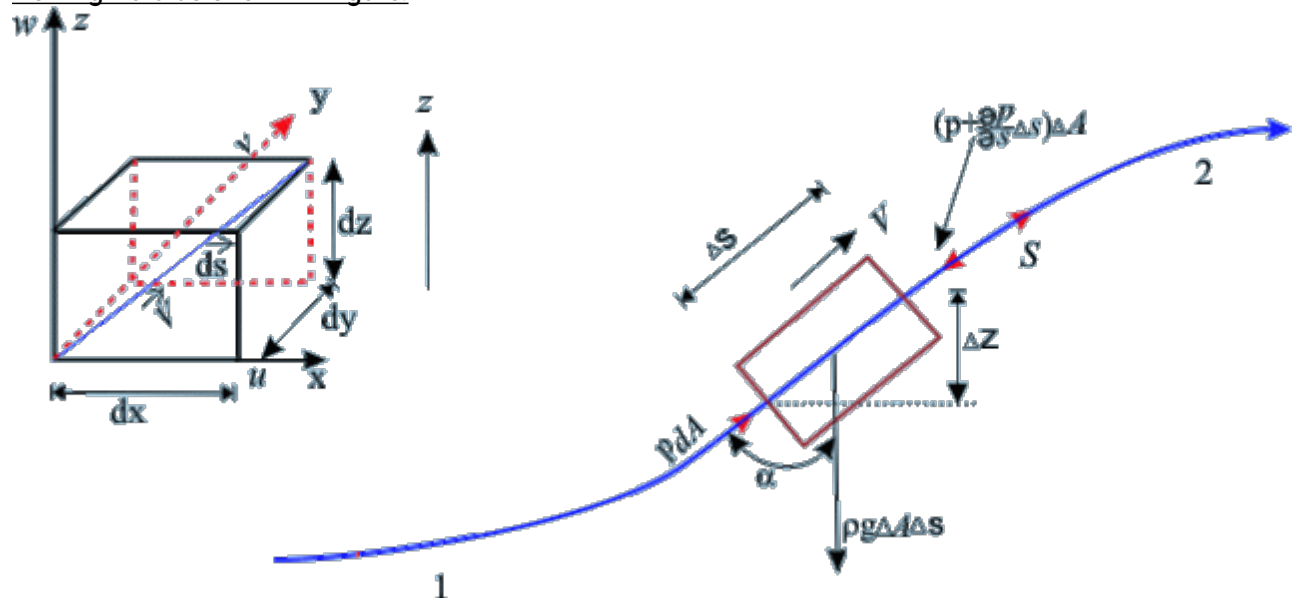
The Euler's equation for steady flow of an ideal fluid along a streamline is a relation between the velocity, pressure and density of a moving fluid. It is based on the Newton's Second Law of Motion. The integration of the equation gives Bernoulli's equation in the form of energy per unit weight of the following fluid.

It is based on the following assumptions:

- The fluid is non-viscous (i.e., the frictional losses are zero).
- The fluid is homogeneous and incompressible (i.e., mass density of the fluid is constant).
- The flow is continuous, steady and along the streamline.
- The velocity of the flow is uniform over the section.
- No energy or force (except gravity and pressure forces) is involved in the flow.

Derivation Of Equation

Let us consider a steady flow of an ideal fluid along a streamline and small element AB of the flowing fluid as shown in figure.



Let,

- dA = Cross-sectional area of the fluid element
- ds = Length of the fluid element
- dW = Weight of the fluid element
- P = Pressure on the element at A
- $P + dP$ = Pressure on the element at B
- v = velocity of the fluid element

The external forces tending to accelerate the fluid element in the direction of the streamline

$$\begin{aligned}
 &= P.dA - (P + dP)dA \\
 &= -dP.dA
 \end{aligned} \tag{1}$$

We also know that the weight of the fluid element,

$$dW = \rho g.dA.ds$$

The component of the weight of the fluid element in the direction of flow,

$$\begin{aligned}
 &= \rho g.dA.ds \cos \theta \\
 &= \rho g.dA.ds \frac{dz}{ds} \\
 &= \rho g.dA.dz
 \end{aligned} \tag{2}$$

∴ Mass of the fluid element = $\rho.dA.ds$

The acceleration of the fluid element

$$\frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = v \frac{dv}{ds} \tag{3}$$

Now, as per Newton's second law of motion, we know that Force = Mass * Acceleration

$$\Rightarrow (-dp.dA) - (\rho g.dA.dz) = \rho.dA.ds \times v.\frac{dv}{ds}$$

Dividing both sides by $(-\rho dA)$

$$\Rightarrow \frac{dP}{\rho} + g.dz = -v.dv$$

or,

$$\Rightarrow \frac{dP}{\rho} + g.dz + v.dv = 0 \tag{4}$$

This is the required Euler's equation for motion as in the form of a differential equation. Integrating the above equation,

$$\frac{1}{\rho} \int dP + \int g.dz + \int v.dv = Constant$$

$$\Rightarrow \frac{P}{\rho} + gz + \frac{v^2}{2} = Constant$$

$$\Rightarrow P + wz + \frac{wv^2}{2g} = Constant$$

$$\Rightarrow \frac{P}{w} + Z + \frac{v^2}{2g} = Constant$$

or in other words,

$$\frac{P_1}{w_1} + Z_1 + \frac{v_1^2}{2g} = \frac{P_2}{w_2} + Z_2 + \frac{v_2^2}{2g} \tag{5}$$

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

(5)

LESSON 2 BERNOULLI EQUATION

The Bernoulli equation shown in (6) above is applicable to network flow through a pipe. The various components of heads e.g pressure, velocity and potential all contribute to the total head expressed as H. The model is an additive model in which each of the components can be treated individually and also as a group if need be. Generally analysis is done for different points, e.g point 1 on the outlet of a reservoir tank to point 2 at the outlet nozzle of water pump stand. And we know that the *total energy per unit weight* or the *total head* does not change i.e it is **constant**, but we need to determine this value using the Bernoulli equation.

CASE 1 – When the Pipe nozzles are closed (Figure 2)

In figure 2 above, the *total head* line is shown, and piezometers are attached at points along the pipe, and measured when the pipe nozzle are closed i.e at point of zero velocity. (Piezometers are simply open ended vertical tubes filled with the same liquid whose pressure they are measuring). The level in the piezometer is the *pressure head as shown in bernoulli equation*.

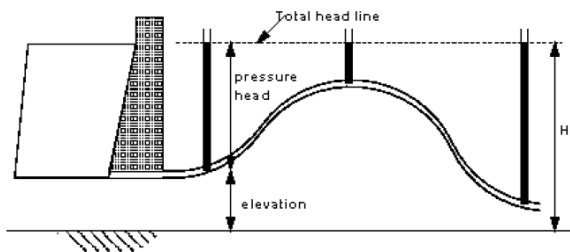


Figure 2: Piezometer Level when the Nozzles are Closed.

Therefore the expected total head for this case 1 is

$$\frac{p}{\rho g} + z = H$$

(6)

CASE 2 – When the Pipe nozzles are opened and flow is allowed (Figure 3)

With the nozzles opened, and water flowing with velocity = u , there is the tendency for the pressure to fall while the velocity increases since the total head will remain the same.

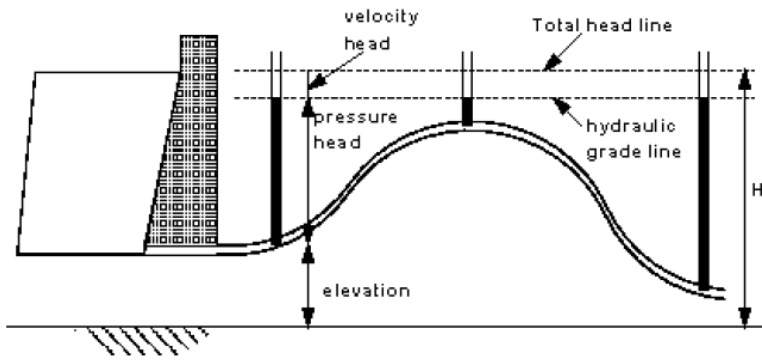


Figure 3: Piezometer level when the nozzles are opened.

Therefore the expected total head for this case 2 is

$$\frac{p}{\rho g} + \frac{u^2}{2g} + z = H \quad (7)$$

Therefore from figure 3, the pressure head reduces by an amount equivalent to the velocity head. Since the pipe is of constant diameter, the velocity therefore will be constant along the pipe so the velocity head and is represented graphically by the horizontal line shown. (this line is known as the *hydraulic grade line*).

CASE 3 – Head Loss Computation (Figure 4)

There are generally energy losses due to friction in a real world order, and must be taken into account. How would the pressure and hydraulic grade lines change with friction? Going back to the constant diameter pipe, we would have a pressure situation like this shown below.

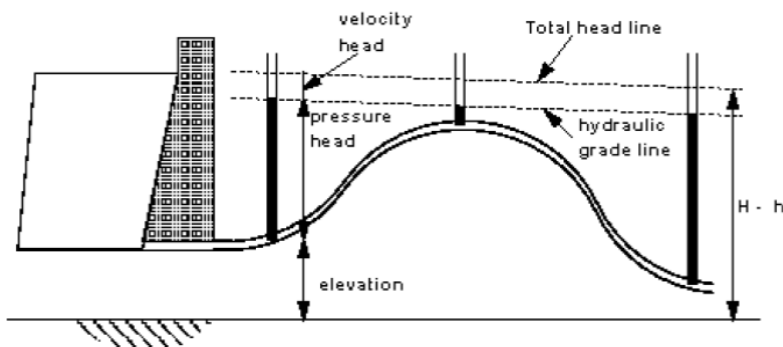


Figure 4: Head losses due to friction

Since head is lost, the total head will therefore not remain constant again.

LESSON 3 Continuity Equation

Since water flows through pipes of varying configuration and size, there is every need to apply the continuity equation.

- Matter cannot be created or destroyed - (it is simply changed in to a different form of matter). This principle is known as the *conservation of mass* and we use it in the analysis of flowing fluids. The principle is applied to fixed volumes, known as control volumes (or

surfaces), like that in the figure 5:

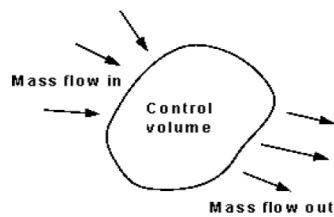


Figure 5: Control Volume

For any control volume the principle of *conservation of mass* says

For **steady flow** there is no increase in the mass within the control volume, so

For steady flow

$$\text{Mass entering per unit time} = \text{Mass leaving per unit time}$$

Mass entering per unit time at end 1 = mass leaving per unit time at end 2 =

$$\rho_1 \delta A_1 u_1 = \rho_2 \delta A_2 u_2 \quad (8)$$

Assuming a constant density,

$$A_1 u_1 = A_2 u_2 = Q \quad (9)$$

This is the continuity equation useful for the computation of flow rate Q . the flow rate across different sections is expected to be the same,

LESSON 4: Tutorial Examples

Example 1: On a circular conduit there are different diameters: diameter $D_1 = 2 \text{ m}$ changes into $D_2 = 3 \text{ m}$. The velocity in the entrance profile was measured: $v_1 = 3 \text{ ms}^{-1}$. Calculate the discharge and mean velocity at the outlet profile (see fig. 1). Determine also type of flow in both conduit profiles (whether the flow is laminar or turbulent) – temperature of water $T = 12^\circ \text{ C}$

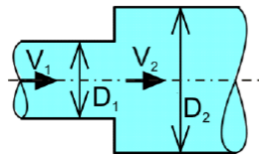


Figure 6: Flow through a pipe network

$$Q = v_1 \cdot S_1 = v_1 \cdot \frac{\pi \cdot D_1^2}{4} = 3 \cdot \frac{\pi \cdot 2^2}{4} = 9,425 \text{ m}^3 \text{ s}^{-1}$$

$$v_2 = \frac{Q}{S_2} = \frac{Q}{\pi \cdot D_2^2 / 4} = \frac{9,425}{\pi \cdot 3^2 / 4} = 1,333 \text{ ms}^{-1}$$

$$Re = \frac{v \cdot D}{\nu}$$

To determine type of flow in conduit, the Reynolds number will be determined considering the following conditions:

For laminar flow: $2320 < Re$.

For turbulent flow $2320 > Re$.

Kinematic viscosity of water of 12°C (Table attached)

$$Re_1 = \frac{v_1 \cdot D_1}{\nu} = \frac{3.2}{1,24 \cdot 10^{-6}} = 4838,1 \cdot 10^3 > 2320 \text{ turbulent flow}$$

$$Re_2 = \frac{v_2 \cdot D_2}{\nu} = \frac{1,333,3}{1,24 \cdot 10^{-6}} = 3225 \cdot 10^3 > 2320 \text{ turbulent flow}$$

Example 2: A horizontal pipeline is attached to the wall of reservoir (see figure 7). The pipeline has different profiles. The water level in the upper reservoir is in the height $H = 1.5 \text{ m}$ above the pipeline axis. From the lower end of the pipeline water flows out to the open space. Diameters and lengths of pipeline reaches are: $D_1 = 0.24 \text{ m}$, $L_1 = 3 \text{ m}$, $D_2 = 0.1 \text{ m}$, $L_2 = 1 \text{ m}$, $D_3 = 0.12 \text{ m}$, $L_3 = 2 \text{ m}$. Calculate discharge in the pipeline and draw the course of energy line (EL) and pressure line (PL).

Resolve the problem:

a) Neglecting losses (i.e. consider the liquid to be ideal)

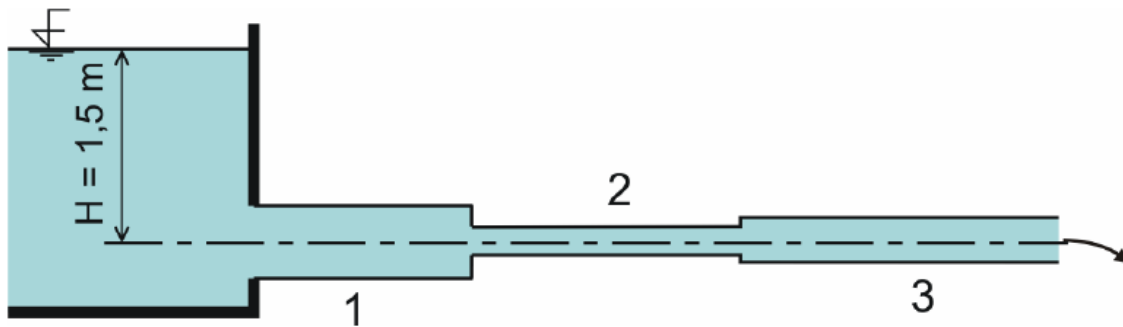


Figure 7:

Bernoulli equation (BE) and **continuity equation** will be used to solve the problem. Using BE to calculate discharge, it will be the most convenient to state the datum (reference) level at the axis of the horizontal pipe, and to write then BE for the upper water level (profile 0 pressure on the level is known - p_a), and for the centre of outlet profile 3 (its position, referred to the datum level, is $h = 0$). The upper reservoir can be considered to be large (its dimensions are not mentioned because they have no importance) and the effect of approach velocity u_n can be therefore neglected (i.e. $u_n = 0$).

$$\text{BE } 0 - 3: \quad H + \frac{p_a}{\rho g} + \frac{v_n^2}{2g} = 0 + \frac{p_a}{\rho g} + \frac{v_3^2}{2g}$$

Arranging this equation, mean velocity in outlet profile can be calculated

$$v_3 = \sqrt{2gH} = \sqrt{2 \cdot 9,81 \cdot 1,5} = 5,42 \text{ m} \cdot \text{s}^{-1}$$

Mass discharge is calculated from **continuity equation**

$$Q = v_3 S_3 = 5,42 \cdot \frac{\pi \cdot 0,12^2}{4} = 0,061 \text{ m}^3 \cdot \text{s}^{-1}$$

Now it is possible to use continuity equation again, to calculate velocities in other profiles and to calculate thus also corresponding velocity heads.

$$v_1 = \frac{Q}{S_1} = \frac{4 \cdot 0,061}{\pi \cdot 0,24^2} = 1,356 \text{ m} \cdot \text{s}^{-1},$$

$$\frac{v_1^2}{2g} = \frac{1,356^2}{19,62} = 0,094 \text{ m}$$

$$v_2 = \frac{Q}{S_2} = \frac{4 \cdot 0,061}{\pi \cdot 0,1^2} = 7,812 \text{ m} \cdot \text{s}^{-1}$$

$$\frac{v_2^2}{2g} = \frac{7,812^2}{19,62} = 3,110 \text{ m}$$

As diameters in the single reaches of the pipeline are constant, also velocities are constant in these reaches. The energy line and pressure line will therefore be parallel and, because of considering the liquid to be ideal, they will be horizontal. Then the BE can be written for 2 profiles - the first one situated at the water level of the upper reservoir, the second one situated e.g. in the middle of length of the first reach of pipe (static pressure there is still unknown):

$$\text{BE 0 - 1:} \quad H + \frac{p_a}{\rho g} + 0 = 0 + \frac{p_{1s}}{\rho g} + \frac{v_1^2}{2g}$$

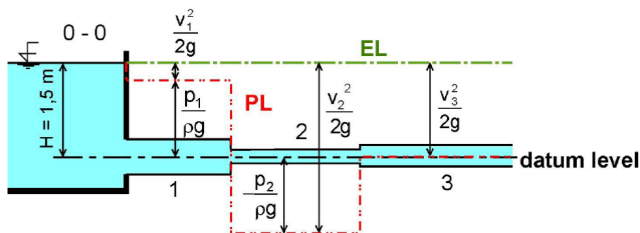
after its rearrangement:

$$\frac{p_1}{\rho g} = \frac{p_{1s} - p_a}{\rho g} = H - \frac{v_1^2}{2g} = 1,5 - 0,094 = 1,406 \text{ m} \quad (\text{overpressure head})$$

Similarly, for the second reach of the pipeline:

$$\frac{p_2}{\rho g} = \frac{p_{2s} - p_a}{\rho g} = H - \frac{v_2^2}{2g} = 1,5 - 3,110 = -1,61 \text{ m} \quad (\text{underpressure head})$$

In the outlet profile there is the atmospheric pressure which, in this case, will be also in the whole length of the last pipe reach. Both velocities and pressures are constant in single reaches of pipeline, energy and pressure lines are therefore parallel, as mentioned. In profiles, where the pipe diameters change, the pressure line changes its position suddenly. Energy line and pressure line are seen on figure 8.



LESSON 5: MODELLING – Bulkingham pi Theorem

This method is illustrated with the drag force of a ship floating on water. There are generally 6 quantities namely: Drag (D), Length (L), Pressure (P), Viscosity (V), Velocity (u) and Gravity (G). These are the n no of quantity. There are also the M no of dimensions namely: Mass (M), length (L), and Time (T). These quantities can be reduced to (n-m) independent dimensionless groups such as Re and Fr.

Determining Dimensionless quantities

$$A_1 = f(A_2, A_3, A_4, A_5, \dots, A_n) \quad (10)$$

A_1, \dots, A_n are the quantities of Function sign.

Steps

- 1 Forming the π group.

For each of the π group, take m of the quantities as the repeating variables, and A_x known as M repeating variables, and one of the other remaining variables. A_1 . Experience shows

which quantity makes the best repeating variables. The π group in general form could then be:

$$\pi_1 = A_1^{x_1} A_2^{y_1} A_3^{z_1} A_4$$

$$\pi_2 = A_1^{x_2} A_2^{y_2} A_3^{z_2} A_5$$

$$\pi_3 = A_1^{x_{n-m}} A_2^{y_{n-m}} A_3^{z_{n-m}} A_n$$

- Set up the π group for MLT system, $m=3$, i.e MLT, and choose A_1 , A_2 , and A_3 as repeating variables.

Using the bulkingham pi theorem on drag force

$$0 = f(D, L, \rho, \mu, V, g)$$

$M = 3$, and $n = 6$.

So there will be $n-m = 3$ π group.

- Select ρ, μ , and L as repeating variable (RV) and leaving D , U , and G as the non repeating variables.

Note: If the analysis is not workable, new repeating variables can always be selected.

$$\pi_1 = \rho^{x_1} \mu^{y_1} L^{z_1} D$$

$$\pi_2 = \rho^{x_2} \mu^{y_2} L^{z_2} \mu$$

$$\pi_3 = \rho^{x_3} \mu^{y_3} L^{z_3} g$$

- Which are all dimensionless quantities of $M^0 L^0 T^0$

$$M^0 L^0 T^0 = \left(\frac{M}{L^3}\right)^{x_1} \left(\frac{L}{T}\right)^{y_1} (L)^{z_1} D$$

Expanding and collecting like terms we can solve the exponents

$$M: 0 = X_1 + 1 = X_1 = -1$$

$$T: 0 = Y_1 - 2 = Y_1 = -2$$

$$L: 0 = -3(X_1) + Y_1 + Z_1 + 1 = Z_1 = -2$$

The exponents are $X_1, Y_1, Z_1, -1, -2$, and -2 .

The first dimensionless group is obtained by substituting the following back to the equation.

Repeating similar procedure to determine the other groups.

LESSON 6; Dimensional Analysis and Similitude

Wind Force on a Van

Suppose you are driving a van down a highway with gusty winds. How does the speed of your vehicle affect the wind force you are experiencing?

The force F of the wind on the van is certainly affected by the speed v of the van and the surface area A of the van directly exposed to the wind's direction. Thus, we might hypothesize that the force is proportional to some power of the speed times some power of the surface area; that is,

$$F = k v^a A^b$$

The dimensions of the variables are as shown



Variable	F	k	v	A
Dimension	MLT^{-2}	$M^0L^0T^0$	LT^{-1}	L^2

Hence the above equation becomes

$$MLT^{-2} = (M^0L^0T^0)(LT^{-1})^a(L^2)^b$$

This last equation cannot be correct because the dimension M for mass does not enter into the right-hand side with nonzero exponent. This can be resolved by including the density factor. If we include the density (ρ) as a factor, then refine our model becomes

$$F = kv^a A^b \rho^c$$

Therefore the equation becomes

$$MLT^{-2} = (M^0L^0T^0)(LT^{-1})^a(L^2)^b(ML^{-3})^c$$

Equating the exponents on both sides of this last equation leads to the system of linear equations:

$$\left. \begin{array}{rcl} & c & = 1 \\ a + 2b - 3c & = & 1 \\ -a & = & -2 \end{array} \right\}$$

Solution of the system (14.8) give $a = 2$, $b = 1$, and $c = 1$. When substituting these values give the model

$$F = kv^2 A \rho$$

Therefore assuming density to be constant and that:

$$F \propto Av^2$$

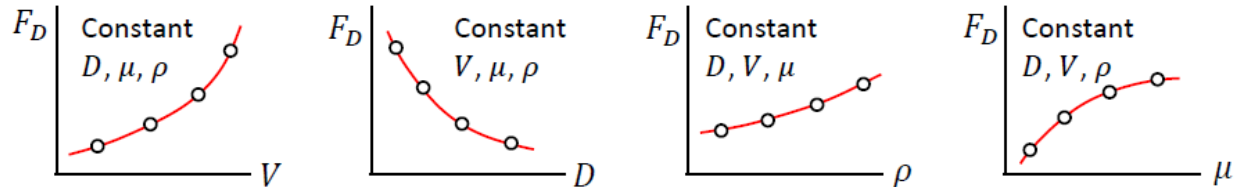
giving $F = k_1 Av^2$. However, k_1 does have a dimension associated with it and is called a **dimensional constant**. In particular, the dimension of k_1 is

$$\frac{MLT^{-2}}{L^2(L^2T^{-2})} = ML^{-3}$$

LESSON 7: Comparing Dimensional Analysis and Experimental Approach

For a controlled experiment using the Drag Force of a Ship as illustrated above, the following set of experiments will be done.

- Fix D , μ , and ρ . Change V and measure F_D .
- Fix D , μ , and ρ . Change V and measure F_D .
- Fix D , μ , and ρ . Change V and measure F_D .
- Fix D , μ , and ρ . Change V and measure F_D .

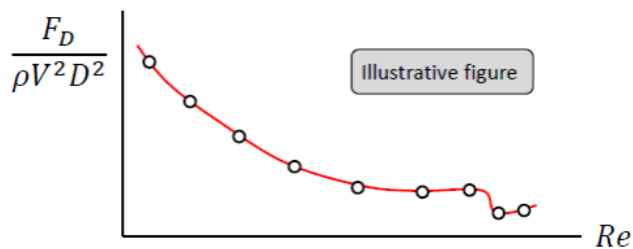


Therefore We need to perform too many experiments.

It is possible to simplify the dependency of drag force on other variables by using nondimensional (unitless) parameters.

$$\frac{F_D}{\rho V^2 D^2} = f_1 \left(\frac{\rho V D}{\mu} \right)$$

Nondimensional drag force (Drag coefficient)
Nondimensional Reynolds number (Re)



To find this new relation, we only need to change the Reynolds number. We can do it in any way we want, e.g. the simplest way is to change the speed of air flow in a wind tunnel.

This eases the stress of conducting four different experiments to determine Drag force.

Summary of the Basic steps in dimensional analysis is thus:

- **Step 1 :** List all the dimensional variables involved in the problem. E.g no of dimensional variables, basic units of dimensions, and the dimensionless pi groups.
- **Step 2 :** Express each of the variables in terms of basic dimensions, which of L : length, T : time, M : mass. E.g $velocity = \frac{L}{T}$
 Variables having only L in their dimension are called **geometric variables**.
 Variables having only T or both L and T are called **kinematic variables**.
 Variables having M in their dimension are called **dynamic variables**.
- **Step 3 :** Determine the **repeating variables** that are allowed to appear in more than one Pi group. There should be $n-1$ many repeating variables.
 If L is a primary dimension of the problem, we should select one geometric variable as a repeating variable.

If ρ is a primary dimension of the problem, we should select one kinematic variable as a repeating variable.

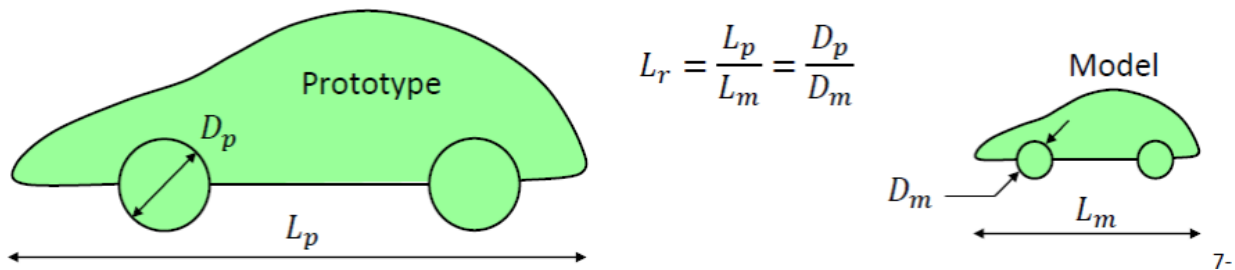
If μ is a primary dimension of the problem, we should select one dynamic variable as a repeating variable.

LESSON 8: Model and Prototype

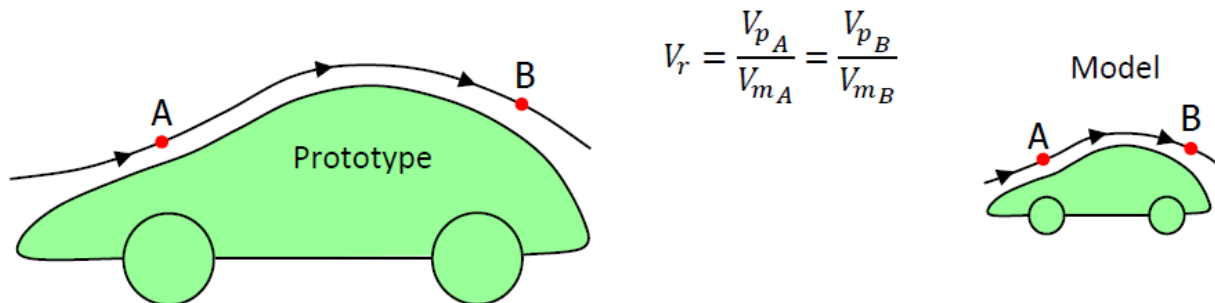
In experimental fluid mechanics we sometimes can not work with real sized objects, known as **prototypes**. Instead we use scaled down (or up) versions of them, called **models**.

Three Basic Laws of Similitude

Geometric similarity : Model and prototype must be the same in shape, but can be different in size. All linear dimensions of the model be related to corresponding dimensions of the prototype by a constant **length ratio**, L_r .



2. Kinematic similarity : Model and prototype flow fields are kinematically similar if the velocities at corresponding points are the same in direction and differ only by a constant factor of **velocity ratio**, V_r .



3. Dynamic similarity : Two flow fields should have force distributions such that identical types of forces are parallel and are related in magnitude by a constant factor of **force ratio**.

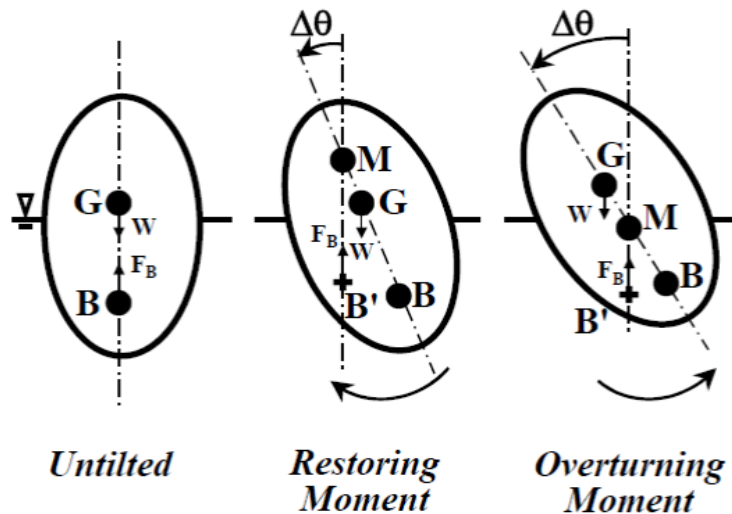
If a certain type of force, e.g. compressibility force, is highly dominant in the prototype flow, it should also be dominant in the model flow. How suitable would it be to use a water tunnel to study the aerodynamic forces acting on a supersonic missile ? If a certain type of force, e.g. surface tension force, is negligibly small in the prototype flow, it should also be small in the model flow. How suitable would it be to use a very light and very small model to test the forces acting on a ship ? To establish dynamic similarity we need to determine the important forces of the prototype flow and make sure that the nondimensional numbers related to those forces are the same in prototype and model flows.

LESSON 9: STABILITY OF FLOATING BODIES

A floating body is **STABLE** if, when it is displaced, it returns to equilibrium.

A floating body is UNSTABLE if, when it is displaced, it moves to a new equilibrium.

Consider a floating body tilted by an angle $\Delta\theta$, as shown below. For the untilted body the point G is the centre of gravity of the body where the body weight, W , acts. The point B is the centre of buoyancy (the centroid of the displaced volume of fluid) where the upward buoyancy force, F_B , acts.



When the body is tilted the centre of buoyancy moves to a new position, B' , because the shape of the displaced volume changes. A new point, M , may be defined, called the METACENTRE. This is the point where a vertical line drawn upwards from the new centre of buoyancy, B' , of the tilted body intersects the line of symmetry of the body. The buoyancy force, F_B , now acts through B' .

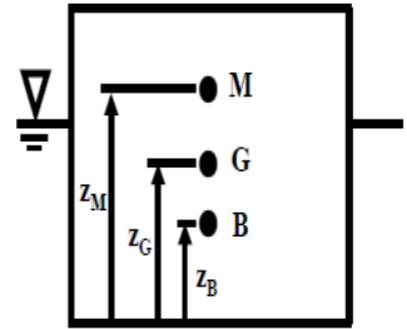
From the centre diagram in the figure we can see that W and F_B give a RESTORING MOMENT that rotates the body back to its untilted position. From the right hand diagram in the figure we can see that W and F_B give an OVERTURNING MOMENT that rotates the body even further in the tilted direction.

Hence, we can say; if the metacentre, M , lies above the centre of gravity, G , then the body is stable. In other words the METACENTRIC HEIGHT, MG , is positive ($MG = z_M - z_G > 0$). If the metacentre, M , lies below the centre of gravity, G , then the body is unstable. In other words the metacentric height, MG , is negative ($MG < 0$).

The metacentric height, MG , is given by

$$MG = MB - GB \quad \text{or} \quad MG = \frac{I}{V_s} - GB$$

where I is the 2nd moment of area of the plan section of the body where it cuts the waterline (this is the solid plane surface you'd see if you cut horizontally through a solid body at the water surface lifted the top part up and looked at the bottom of it!), V_s is the submerged volume (i.e. volume of fluid displaced) and GB is the distance between the centre of gravity and the centre of buoyancy ($= z_G - z_B$).



Strategy for solving buoyancy problems

- (1) From geometry of body and density of fluid and body equate; Weight of displaced fluid = Total weight of body. This gives the depth of immersion of the body or the weight of the body, whichever is unknown.
- (2) To assess stability, first find the location of the centre of gravity G of the body.
- (3) Then, find the location of the centre of buoyancy B (centroid of displaced volume). For a regularly shaped body this will be at half the height of the immersed portion of the body.
- (4) Calculate the distance GB .
- (5) Calculate MB , using $MB = I / V_s$. Note $I = \pi D^4/64$ for a circular section body and $bd^3/12$ for a rectangular section body (D is diameter, b and d are the sides of the rectangle).
- (6) Calculate metacentric height, $MG (= z_M - z_G)$, from $MG = MB - GB$. If $MG > 0$ then body is stable. If $MG < 0$ then body is unstable.

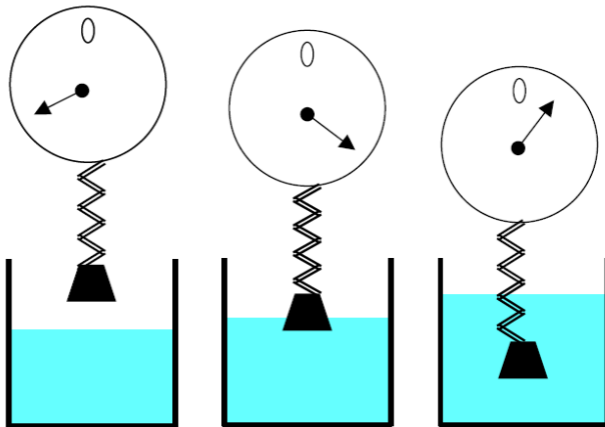
LESSON 10: Buoyancy

PRINCIPLES

- Buoyance: It is an upward thrust exerted by a liquid on a body floating or immersed in it. It is equal to the weight of the liquid displaced by the body.
- **Archimedes Principle:** Whenever a body immersed wholly or partially in a liquid it is acted upon by a vertical upward liquid thrust equal to the weight of the liquid displaced.
- Center of Buoyancy: It is the point at which the buoyancy or upward liquid thrust acts. The centre of buoyancy will be the centre of gravity of the fluid displaced.

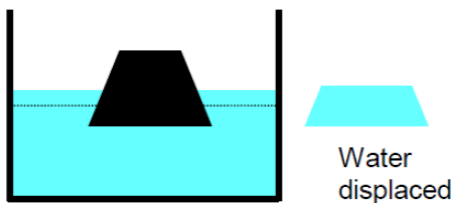
When a solid object is wholly or partly immersed in a fluid, the fluid molecules are continually striking the submerged surface of the object. The forces due to these impacts can be combined into a single force the **buoyant force**. The immersed object will be "lighter" i.e. It will be buoyed up by an

amount equal to the weight of the fluid it displaces.



Floating: partially submerged

- Weight of object < weight of fluid that can be displaced by object.
- Volume of displaced water < volume of object.
- Weight of liquid displaced by partially submerged object = weight of object



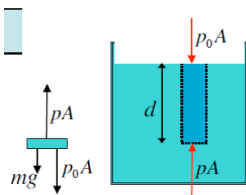
Floating: fully submerged

- Weight of object = weight of fluid displaced by object.
- Volume of displaced water = volume of object

Sinks

- Weight of object > weight of fluid displaced by object.
- Volume of displaced water = volume of object.

Pressure in a liquid



Atmospheric pressure p_0 pushes down with force of p_0A . The weight of the column pushes down with force mg . For a liquid with density ρ , $m = \rho V = \rho A d$.

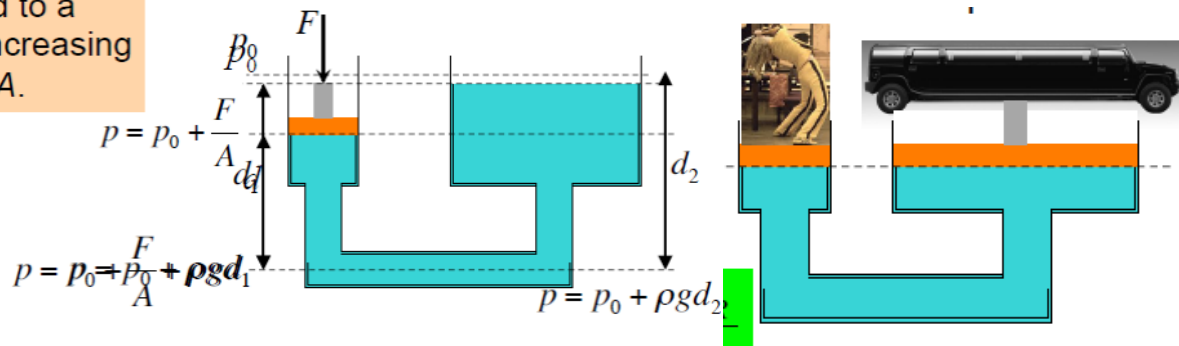
Because the liquid is in static equilibrium, the up

$$pA = p_0A + \rho A d g \quad \text{so} \quad p = p_0 + \rho g d \quad (\text{hydrostatic pressure})$$

Some rules for pressure

Pascal's law: A pressure change at one point in an incompressible fluid appears undiminished at all points in the fluid. A force F is applied to a piston of area A , increasing the pressure by F/A .

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/A.



LESSON 11 &12

Tutorial examples using the manual Titled - Buoyancy and Flotation Tutorial Example. This material is a compilation of various examples.